DETERMINANTS.

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§ 1. If we have the two simultaneous linear equations

$$a_1x + b_1y = c_1 \qquad a_2x + b_2y = c_2$$

we can, performing the elimination by any of the common methods, obtain values for the variables in terms of the coefficients: thus we find

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \text{ and } y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

These results may be exhibited more compactly thus

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},$$

where the symbol $\left|\quad\right|$ means the *Determinant* of the four quantities enclosed within the vertical lines, that is to say, the algebraic sum of their combinations in sets of two and two, with the condition that each combination contains one, and only one, of the quantities in each *row* or *column*: and that the signs are determined by the following rule. The *Diagonal* of the Determinant, viz., the combination got by reading from the left-hand upper corner to the right-hand lower corner, is affected with the sign *plus*: and then the sign *plus* or *minus* is affixed to each of the remaining combinations, according as it may be derived from the Diagonal by an *even* or by an *odd* number of interchanges among the suffixes attached to the quantities a_1, b_1 &c.

The truth of this rule is rendered evident, in the case before us, by an inspection of the Determinant in its expanded form.

§ 2. Now consider the three equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3. \end{aligned}$$

From these we obtain, by the common methods of elimination,

$$x = \frac{d_1b_2c_3 - d_1b_3c_2 + d_2b_3c_1 - d_2b_1c_3 + d_3b_1c_2 - d_3b_2c_1}{a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1}$$

and similar expressions for the values of y and z.

The denominator of this value of x (and also of y and z) is a determinant, that is, may be symbolically exhibited as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

And the numerator may be written

$$\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$ which may be regarded as having been derived from the Determinant constituting the denominator by erasing the column composed of the coefficients of x and replacing it by that composed of the absolute terms. And in a precisely similar manner the numerators of the values of y and z may be exhibited as Determinants.

§ 3. These illustrations will suffice to give the student a general notion of the nature of Determinants as they occur in the solution of systems of simultaneous linear equations. In the development of the subject however, even in the elementary manner here proposed, it is desirable to give the definitions and establish the properties without direct reference to equations between variables: and this we proceed to do.

§ 4. *Definitions.* 1°. If there be n^2 quantities arrayed in a square of n rows and n columns then the sum with the proper signs (as fixed by the rule given in § 1,) of all possible products of these n^2 quantities in sets of n, one quantity, and only one, being taken from each horizontal row and from each vertical column, is called the *Determinant* of these n^2 quantities. And the Determinant is said to be of the n^{th} order.

2°. Each of the n^2 quantities is called an *element*, and each of the products of n elements, a *constituent* of the Determinant. The Determinant is represented by enclosing the n^2 quantities within two vertical lines, as in the examples already given.

3°. If there be mn quantities arrayed in m rows and n columns, and any number of rows and as many columns be selected, the square thus formed is called a *Minor* of the given array: and if $n < m$ the minors of the n^{th} degree are called *principal minors.* If we desire to represent the Determinants of all the principal minors of the mn quantities we will enclose the array in *double* vertical lines;

e. g. $\begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{Vmatrix}$ denotes the three Determinants $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$, $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ and $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

§ 5. General Properties of Determinants.

1°. *The value of a Determinant is not altered if the successive rows are changed into successive columns.* Thus

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

This principle is evident, generally, from the law of formation (§ 1) being perfectly symmetrical with regard to rows and columns.

2°. *If any two rows (or columns) be interchanged, the sign of the Determinant will be changed.* For this change is evidently equivalent to a single permutation among the suffixes (or letters) and this by the law of formation causes a change of sign.

3°. *If two rows (or columns) are identical, the Determinant vanishes.* For if we interchange these rows we ought to have a change of sign by the preceding property: but the interchange of two identical rows can produce no change in the value of the Determinant: in other words, the Determinant is equal to itself with its sign changed, but this can only be when it is equal to zero.

4°. *If every element in any row (or column) be multiplied by the same factor, the Determinant is multiplied by that factor.* This appears from the fact that every constituent of the Determinant contains one and but one element from the same row or the same column.

5°. *If the elements in one row (or colamn) be like multiples of those of another row (or column) the Deteminant vanishes.* (3° and 4°.)

E. g.
$$\begin{vmatrix} ka_2 & kb_2 & kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

6°. A Determinant of the n^{th} order may be expressed in terms of Determinants of the $(n - 1)^{\text{th}}$ order. E. g.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$
$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

When the Determinant is written in this form, the factors multiplying a_1, a_2 &c., or a_1, b_1 &c., are recognised as the minors obtained from the given Determinant by erasing in succession the columns and rows containing a_1, a_2 &c., or a_1, b_1 &c. Let the Minors corresponding to a_1, a_2 &c., be represented by A_1, A_2 &c.: those corresponding to b_1, b_2 &c., by B_1, B_2 &c.: then the original determinant may be briefly written

$$a_1 A_1 - b_1 B_1 + c_1 C_1 \text{ \&c.},$$

or
$$a_1 A_1 - a_2 A_2 + a_3 A_3 \text{ \&c.}$$

7°. *If all the elements but one of any row (or column) of a Determinant vanish, the order of the Determinant is reduced by one.* For if a_2, a_3 &c., in the Determinant last above written all vanish the Determinant becomes $a_1 A_1$, and A_1 is of the $(n - 1)^{\text{th}}$ order.

8°. Hence any Determinant may be exhibited in the form of one of any higher order:

e. g.
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ x & a_1 & b_1 & c_1 \\ y & a_2 & b_2 & c_2 \\ z & a_3 & b_3 & c_3 \end{vmatrix}$$
where x, y and z have any values; and this process may be extended indefinitely.

9°. If every element in any row (or column) be a sum, the Determinant is resolvable into the sum of others. Thus, if in the Determinant $a_1A_1 - b_1B_1 + c_1C_1$ &c., we write $a_1 + k$ for a_1, $b_1 + l$ for b_1, $c_1 + m$ for c_1 &c., the Determinant becomes $(a_1 + k)A_1 - (b_1 + l)B_1 + (c_1 + m)C_1$ &c., and this can be written $(a_1A_1 - b_1B_1 + c_1C_1$ &c.,$) + (kA_1 - lB_1 + mC_1$ &c.). Similarly, if the elements in any one row (or column) were each the sum of several numbers, the Determinant could be presented as the sum of a like number of Determinants.

10°. *If the elements of one row (or column) are respectively equal to those of other rows (or columns) multiplied respectively by constant factors, the Determinant vanishes.* For it is the sum of other Determinants which are separately evanescent. E. g.
$$\begin{vmatrix} kb_1 + lc_1, & b_1, & c_1 \\ kb_2 + lc_2, & b_2, & c_2 \\ kb_3 + lc_3, & b_3, & c_3 \end{vmatrix} = \begin{vmatrix} kb_1, & b_1, & c_1 \\ kb_2, & b_2, & c_2 \\ kb_3, & b_3, & c_3 \end{vmatrix} + \begin{vmatrix} lc_1, & b_1, & c_1 \\ lc_2, & b_2, & c_2 \\ lc_3, & b_3, & c_3 \end{vmatrix}$$
and these latter vanish by 5°.

11°. *A Determinant is not altered if we add to each element of any row (or column) the corresponding elements of any other row (or column) multiplied by constant factors.* E. g.
$$\begin{vmatrix} a_1 + kb_1, & b_1, & c_1 \\ a_2 + kb_2, & b_2, & c_2 \\ a_3 + kb_3, & b_3, & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} kb_1 & b_1 & c_1 \\ kb_2 & b_2 & c_2 \\ kb_3 & b_3 & c_3 \end{vmatrix}$$
but the last Determinant vanishes by 5°.

These principles will now be applied to the calculation of the numerical values of the following Determinants.

§ 6. Examples.

1. $$\begin{vmatrix} 1 & 1 & 1 \\ 3 & -1 & 2 \\ 4 & 3 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} - \begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ 4 & -3 \end{vmatrix} \text{ by } 6°,$$
$$= (1 - 6) - (-3 - 8) + (9 + 4) = 19.$$
Or thus, using the principle given in 11°; subtract the elements of the first column from the corresponding elements of the second and third columns: this, without altering the value of the Determinant reduces it to
$$\begin{vmatrix} 1 & 0 & 0 \\ 3 & -4 & -1 \\ 4 & -1 & -5 \end{vmatrix} \text{ which by 7° is equivalent to } \begin{vmatrix} -4 & -1 \\ -1 & -5 \end{vmatrix} \text{ or } 20 - 1 = 19.$$

2. $$\begin{vmatrix} 3 & 4 & -5 \\ 4 & -5 & 3 \\ 5 & -3 & -4 \end{vmatrix} = 3\begin{vmatrix} -5 & 3 \\ -3 & -4 \end{vmatrix} - 4\begin{vmatrix} 4 & 3 \\ 5 & -4 \end{vmatrix} - 5\begin{vmatrix} 4 & -5 \\ 5 & -3 \end{vmatrix}$$

$$=3(20+9)-4(-16-15)-5(-12+25)=146.$$

3. $$\begin{vmatrix} 32 & 4 & -5 \\ 18 & -5 & 3 \\ 2 & -3 & -4 \end{vmatrix} = 2\begin{vmatrix} 16 & 4 & -5 \\ 9 & -5 & 3 \\ 1 & -3 & -4 \end{vmatrix} = 2\begin{vmatrix} 16 & 52 & 59 \\ 9 & 22 & 39 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 2\begin{vmatrix} 52, & 59 \\ 22, & 39 \end{vmatrix} = 2(2028 - 1298) = 1460.$$

The 3rd is obtained from the 2nd by adding to the 2nd and 3rd columns respectively, the elements of the 1st multiplied by 3 and 4 respectively.

4. $$\begin{vmatrix} 3 & 32 & -5 \\ 4 & 18 & 3 \\ 5 & 2 & -4 \end{vmatrix} = 2\left[3\begin{vmatrix} 9 & 3 \\ 1 & -4 \end{vmatrix} - 16\begin{vmatrix} 4 & 3 \\ 5 & -4 \end{vmatrix} + -5\begin{vmatrix} 4 & 9 \\ 5 & 1 \end{vmatrix}\right]$$

$$= 2[3(-36-3)-16(-16-15)-5(4-45)]$$

$$= 2[-117 + 496 + 205] = 2 \times 584 = 1168.$$

5. $$\begin{vmatrix} 3 & 4 & 32 \\ 4 & -5 & 18 \\ 5 & -3 & 2 \end{vmatrix} = 3\begin{vmatrix} -5 & 18 \\ -3 & 2 \end{vmatrix} - 4\begin{vmatrix} 4 & 18 \\ 5 & 2 \end{vmatrix} + 32\begin{vmatrix} 4 & -5 \\ 5 & -3 \end{vmatrix}$$

$$= 3(-10+54)-4(8-90)+32(-12+25)=876.$$

6. (From Salmon's Lessons on Modern Higher Algebra.)

$$\begin{vmatrix} 9 & 13 & 17 & 4 \\ 18 & 28 & 33 & 8 \\ 30 & 40 & 54 & 13 \\ 24 & 37 & 46 & 11 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 4 \\ 2 & 4 & 1 & 8 \\ 4 & 1 & 2 & 13 \\ 2 & 4 & 2 & 11 \end{vmatrix} = \begin{vmatrix} 1, & 1, & 1, & 1, \\ 2, & 4, & 1, & 1, \\ 4, & 1, & 2, & 6, \\ 2, & 4, & 2, & 3, \end{vmatrix}$$

$$= \begin{vmatrix} 1, & 0, & 0, & 0, \\ 2, & 2, & -1, & -1, \\ 4, & -3, & -2, & 2, \\ 2, & 2, & 0, & 1, \end{vmatrix} = \begin{vmatrix} 2, & -1, & -1, \\ -3, & -2, & 2, \\ 2, & 0, & 1, \end{vmatrix} = \begin{vmatrix} 4, & -1, & -1 \\ -7, & -2, & 2 \\ 0, & 0, & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 4, & -1, \\ -7, & -2, \end{vmatrix} = -8 - 7 = -15.$$

The second Determinant is derived from the first by subtracting from the elements of the first, second and third columns, twice, three times, and four times the corresponding elements of the last column. The remaining steps are very similar.

7. $$\begin{vmatrix} a, & b, & c, \\ c, & a, & b, \\ b, & c, & a, \end{vmatrix} = a\begin{vmatrix} a, & b, \\ c, & a, \end{vmatrix} - b\begin{vmatrix} c, & b, \\ b, & a, \end{vmatrix} + c\begin{vmatrix} c, & a, \\ b, & c, \end{vmatrix}$$

$$= a(a^2-bc)-b(ac-b^2)+c(c^2-ab)=a^3+b^3+c^3-3abc.$$

Moreover, since by 11° the Determinant is unchanged in value if written

$$\begin{vmatrix} a+b+c, & b, & c, \\ c+a+b, & a, & b, \\ b+c+a, & c, & a, \end{vmatrix} = (a+b+c)\begin{vmatrix} 1, & b, & c, \\ 1, & a, & b, \\ 1, & c, & a, \end{vmatrix}$$

it is evident that $a + b + c$ is a factor of $a^3 + b^3 + c^3 - 3abc$.

8. $$\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix} = a\begin{vmatrix} a, & b, & c, \\ d, & a, & b, \\ c, & d, & a, \end{vmatrix} - b\begin{vmatrix} d, & b, & c, \\ c, & a, & b, \\ b, & d, & a, \end{vmatrix}$$

$$+ c\begin{vmatrix} d, & a, & c, \\ c, & d, & b, \\ b, & c, & a, \end{vmatrix} - d\begin{vmatrix} d, & a, & b, \\ c, & d, & a, \\ b, & c, & d, \end{vmatrix} = \left\{ \begin{array}{l} a^4 - b^4 + c^4 - d^4 - 2a^2c^2 \\ + 2b^2d^2 - 4a^2bd + 4b^2ac \\ - 4c^2bd + 4d^2ac, \end{array} \right.$$

and as in Ex. 7, it appears that $a+b+c+d$ is a factor of this Determinant.

9. $$\begin{vmatrix} 0, & c, & b, \\ c, & 0, & a, \\ b, & a, & 0, \end{vmatrix} = - c(0 - ab) + b(ac - 0) = 2abc.$$

§ 7. Applications of Determinants.

If we write the Determinant

$$\begin{vmatrix} a_1 & b_1 \cdots\cdots & n_1 \\ a_2 & b_2 & n_2 \\ \vdots & \vdots & \vdots \\ a_n & b_n & n_n \end{vmatrix}$$

in the form given in 6°, § 5, viz., $a_1A_1 - a_2A_2 + a_3A_3 \ldots \pm a_nA_n$ it is easy to establish the following relations:—

$$b_1A_1 - b_2A_2 \ldots \pm b_nA_n = 0,$$

$$c_1A_1 - c_2A_2 \ldots \pm c_nA_n = 0,$$ &c., for the left-hand members are evidently what the Determinant becomes when b_1, b_2 &c., or c_1, c_2 &c., are written for a_1, a_2 &c., that is to say, when there are two identical rows: but by 3°, § 5, in this case the Determinant vanishes.

§ 8. Let there be given the following system of simultaneous linear equations,

$$\begin{array}{l} a_1x + b_1y + c_1z \quad \ldots\ldots = k_1 \\ a_2x + b_2y + c_2z \quad \ldots\ldots = k_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_nx + b_ny + c_nz \quad \ldots\ldots = k_n \end{array}$$

and let the first be multiplied by A_1, the second by $-A_2$, the third by A_3 &c., and all added: the coefficient of x will be $a_1A_1 - a_2A_2$ &c., and those of the other variables will vanish by virtue of the relations established in the preceding article. Thus we have

$$(a_1A_1 - a_2A_2 \text{ \&c.})x = k_1A_1 - k_2A_2 + k_3A_3 \text{ \&c.}$$

Here we observe that the coefficient of x is the Determinant of the square formed of the coefficients of thevariabless, and the right-hand member is evidently the Determinant of the square got by erasing the column of the coefficients of x and replacing it by that composed of the absolute terms. Let us denote the former by V and the latter by D_1. Then the value of x as above determined is $x = \frac{D_1}{V}$.

Returning to the given equations, and multiplying them by B_1, B_2, B_3 &c., and adding, the coefficients of all the variables except y will vanish, while that of y will be $b_1B_1 - b_2B_2$ &c., which by § 5 is equivalent to the Determinant V; and the sum of the right-hand members will be $k_1B_1 - k_2B_2 + k_3B_3$ &c., and this is the Determinant of the square got by erasing the coefficients of y and replacing them by the corresponding absolute terms: calling this Determinant D_2, we have $V.y = D_2$;

$\therefore \quad y = \frac{D_2}{V}$. Similarly, $z = \frac{D_3}{V}$ and so for the other variables.

Example. Given $\left.\begin{array}{l} 3x + 4y - 5z = 32 \\ 4x - 5y + 3z = 18 \\ 5x - 3y - 4z = 2 \end{array}\right\}$ to find x, y and z.

The Determ nants V, D_1, D_2, and D_3 are those whose values have already been computed in Examples 2, 3, 4 and 5 in § 6. Referring to them we have $x = \frac{D_1}{V} = \frac{1460}{146} = 10$; $y = \frac{D_2}{V} = \frac{1168}{146} = 8$, & $z = \frac{D_3}{V} = \frac{876}{146} = 6$.

§ 9. The following results follow as corollaries from § 8.

1°. If V vanishes the values of the variables become infinite, thus indicating that the proposed equations are *inconsistent.*

2°. If $V = 0$ and the absolute terms also all vanish, the values of the variables as furnished by the method of § 8 assume the form $0 \div 0$. It is practicable however to determine the ratios of $n - 1$ of the unknown quantities to the remaining unknown, thus:—take the three equations

$a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$, $a_3x + b_3y + c_3z = 0$;

we may write them in the form

$$a_1\frac{x}{z} + b_1\frac{y}{z} = -c_1,\ a_2\frac{x}{z} + b_2\frac{y}{z} = -c_2,\ a_3\frac{x}{z} + b_3\frac{y}{z} = -c_3:$$

solving these for the ratios $\frac{x}{z}$ and $\frac{y}{z}$ we have, (by § 8), using the first two equations, the second and third equations, and the first and third equations, respectively;

$$\frac{x}{z} = \frac{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} -c_2 & b_2 \\ -c_3 & b_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} = \frac{\begin{vmatrix} -c_1 & b_1 \\ -c_3 & b_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}}.$$

And similarly $$\frac{y}{z} = \frac{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} a_2 & -c_2 \\ a_3 & -c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} = \frac{\begin{vmatrix} a_1 & -c_1 \\ a_3 & -c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}}.$$

These three forms of solution can be shown to coincide; and the same process may be extended to n equations involving n unknown quantities.

3°. If the absolute terms all vanish and V does not vanish, the only possible solutions are $x = 0$, $y = 0$ &c. Thus it appears that in order that a system of n homogeneous equations in n variables shall have values for the variables other than zero, it is necessary that the Determinant V shall vanish.

4°. If we have n non-homogeneous equations containing $n - 1$ variables $V = 0$ is a necessary condition of consistence and independence; and by combining any $n - 1$ of the equations (as in 2°) the values of the $n - 1$ variables are found.

§ 10. To find the relation which must hold among the coefficients A, B &c., in order that the quadratic function $Ax^2 + By^2 + Cz^2 + 2Dxy + 2Eyz + 2Fxz$ (1) shall break into two factors of the first degree in x, y and z.

The proposed function may be written $(Ax+Dy+Fz)x+(By+Dx+Ez)y+(Cz+Ey+Fx)z$ (2). If the relation between the coefficients is such that, for all values of x, y and z, the linear functions within the parenthesis have some constant ratio, say $p:q:r$, we have $By+Dx+Ez=\frac{q}{p}\left(Ax+Dy+Fz\right)$, and $Cz+Ey+Fx=\frac{r}{p}\left(Ax+Dy+Fz\right)$: then (2) may be written $\left(x+\frac{q}{p}y+\frac{r}{p}z\right)\times(Ax+Dy+Fz)$. This therefore is the condition to be fulfilled: it may be written $\frac{Ax+Dy+Fz}{p}=\frac{By+Dx+Ez}{q}=\frac{Cz+Ey+Fx}{r}$:

and since these relations are to hold regardless of the values of x, y and z, they will hold for these which reduce the numerators to zero: that is, we have

$$Ax + Dy + Fz = 0,$$
$$By + Dx + Ez = 0,$$
$$Cz + Ey + Fx = 0.$$

Eliminating the variables we get

$$\begin{vmatrix} A & D & F \\ D & B & E \\ F & E & C \end{vmatrix} = 0$$

as the necessary relation between the coefficients in order that the function proposed shall break into two factors. This Determinant is, by some writers, called the Discriminant of the proposed function.

This is all that our limits allow on this subject. Students desiring to pursue the subject are recommended to read Baltzer's Treatise on Determinants, Salmon's Lessons on Modern Higher Algebra, Spottiswoode, Todhunter, etc.